# Compact connected spaces via the projective Fraïssé limit constructions

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# Topological graphs and monotone maps

## Definition

A subset $S$ of a topological graph $G$ is disconnected if there are two nonempty closed subsets $P$ and $Q$ of $S$ such that $P \cup Q = S$ and if $a \in P$ and $b \in Q$, then $\langle a, b \rangle \notin E(G)$. A subset $S$ of $G$ is connected if it is not disconnected.

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**Definition**

Let $G$, $H$ be topological graphs. An epimorphism $f \colon G \to H$ is called monotone if for every closed connected subset $Q$ of $H$, $f^{-1}(Q)$ is connected.

# Dendrites

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A dendrite is an arcwise connected, locally connected, hereditarily unicoherent continuum.

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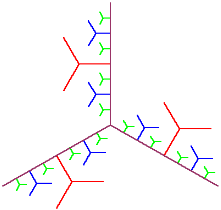
## Theorem (Charatonik-Roe '22+)

*Suppose that $\mathcal{T}$ is a projective Fraïssé class of finite trees with monotone epimorphisms, and $\mathbb{T}$ is the projective Fraïssé limit of $\mathcal{T}$. Then $|\mathbb{T}|$, the topological realization of $\mathbb{T}$, if exists, it is a dendrite.*

# Ważewski dendrites

**Theorem (Charatonik-Roe '22+)**

*The topological realization of the projective Fraïssé limit of the class $\mathcal{T}_M$ of* **all** *finite trees with* **all** *monotone maps, is homeomorphic to the Ważewski dendrite $W_3$.*



# The universal Ważewski dendrite

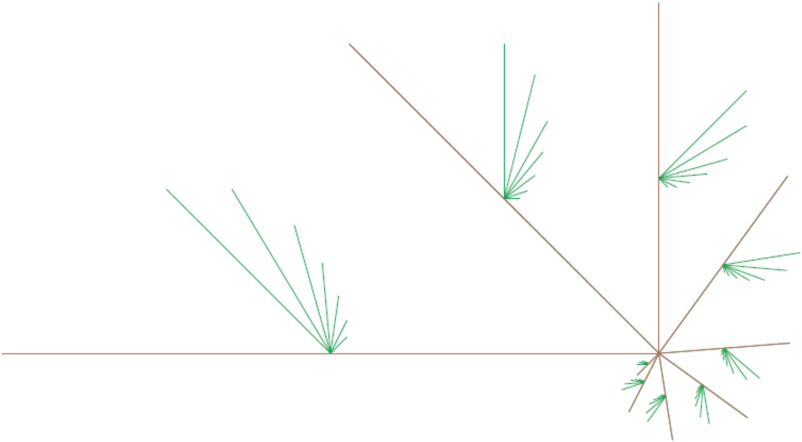


Figure: The universal Ważewski dendrite $W_\omega$

# Ważewski dendrites 2

- For any $P \subseteq \{3, 4, 5, \ldots, \omega\}$ there is a unique Ważewski dendrite, which has ramification points of orders exactly in $P$.
- Codenotti and Kwiatkowska in 2022 considered Fraïssé classes of finite trees with (weakly) coherent monotone epimorphisms, and constructed all Ważewski dendrites as topological realizations of their Fraïssé limits.
- They used uniqueness of Fraïssé limits to give a new proof of a result by Charatonik and Dilks that endpoints of any Ważewski dendrite are countably dense homogeneous.

# Confluent maps

## Definition

- (continua) Let $K, L$ be continua. A continuous map $f : L \to K$ is called confluent if for every subcontinuum $M$ of $K$ and every component $C$ of $f^{-1}(M)$ we have $f(C) = M$.
- (graphs) Let $G$, $H$ be topological graphs. An epimorphism $f : G \to H$ is called confluent if for every closed connected subset $Q$ of $H$ and every component $C$ of $f^{-1}(Q)$ we have $f(C) = Q$.

# More on confluent maps

**Proposition (Charatonik-Roe '22+)**

*Given two finite graphs $G$ and $H$, the following conditions are equivalent for an epimorphism $f : G \to H$:*

1. *$f$ is confluent;*
2. *for every edge $P \in E(H)$ and every component $C$ of $f^{-1}(P)$ there is an edge $E$ in $C$ such that $f(E) = P$.*

# Solenoids

## Definition

A solenoid is a continuum homeomorphic to the inverse limit $\Sigma(\mathbf{p}) = \varprojlim(S^1, f_n)$ of the inverse sequence of unit circles $S^1$ in the complex plane with bonding maps $f_n(z) = z^{p_n}$, where $\mathbf{p} = (p_1, p_2, \dots)$ is a sequence of prime numbers. It is called a $\mathbf{p}$-adic solenoid. The solenoid $\Sigma(2, 2, \dots)$ is known as a dyadic solenoid .

# Solenoids-characterizations

**Theorem**

*Let $X$ be a continuum not homeomorphic to a circle. The following are equivalent:*

1. *$X$ is a solenoid,*
2. *(Hewitt '63) $X$ is homeomorhic to a one-dimensional topological group,*
3. *(Hagopian '77) $X$ is homogeneous and every proper subcontinuum is an arc.*
4. *(Krupski '84) $X$ is circle-like, has the property of Kelley, and contains no local end point,*

# Cycles

## Definition

For $A \in \mathcal{G}$ we will say that $C \subseteq A$ is a cycle in $A$ if $|V(C)| > 2$ and we can enumerate the vertices of $C$ as $(c_0, c_1, \ldots, c_n = c_0)$ in a way that $c_i \neq c_j$ whenever $0 \leq i < j < n$ and $\langle c_i, c_j \rangle \in E(A)$ if and only if $|j - i| \leq 1$.

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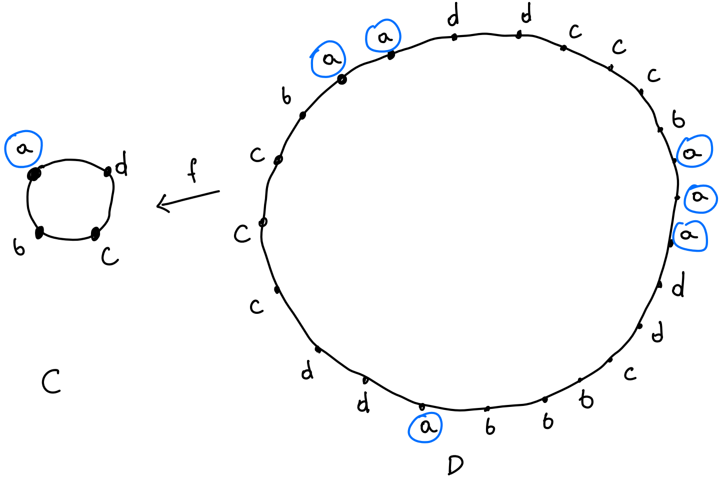
## Definition

The winding number of a wrapping map $f$ is $n$ if for every (equivalently: some) $c \in C$, $f^{-1}(c)$ has exactly $n$ components.

# Wrapping maps



# Fraïssé classes of cycles - warm up

**Proposition**

*Let $P$ be a set of prime numbers and let $\mathcal{D}_P$ be the class of cycles with confluent epimorphisms whose winding numbers are of the form $p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$, where $p_i \in P$ and $n_i \in \mathbb{N}$. Then $\mathcal{D}_P$ is a projective Fraïssé class.*

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## Theorem (Charatonik-K-Roe)

*The $\mathcal{D}_P$ approximates the solenoid, which is projectively universal for all $\mathbf{p}$-adic solenoids, with $\mathbf{p} = (p_1, p_2, p_3, \dots)$, $p_i \in P$.*

# The projective Fraïssé class $\mathcal{G}$ - main example

**Proposition (Charatonik-Roe '22+)**

*The class $\mathcal{G}$ of finite connected graphs with confluent epimorphisms is a projective Fraïssé class.*

- Let $\mathbb{G}$ denote the projective Fraïssé limit. Then $E(\mathbb{G})$ is an equivalence relation with only single and double equivalence classes.
- Let $|\mathbb{G}|$ denote the topological realization. This is a one-dimensional continuum.

# Main Theorem - part 1

**Theorem (Charatonik-K-Roe '22+)**

$|\mathbb{G}|$ *has the following properties:*

1. *it is not homogeneous;*
2. *it is pointwise self-homeomorphic;*
3. *it is an indecomposable continuum;*
4. *all arc components are dense;*
5. *each point is the top of the Cantor fan;*
6. *the pseudo-arc, the universal pseudo-solenoid, and the universal solenoid, embed in it;*
7. *it is hereditarily unicoherent, in particular, the circle $S^1$ does not embed in it.*

# Indecomposable continuum

## Definition

A connected topological graph $G$ is called decomposable if there are two closed connected subgraphs $A$ and $B$ such that $G = A \cup B$, $A \neq G$, and $B \neq G$. It is called indecomposable if it is not decomposable.

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**Theorem**

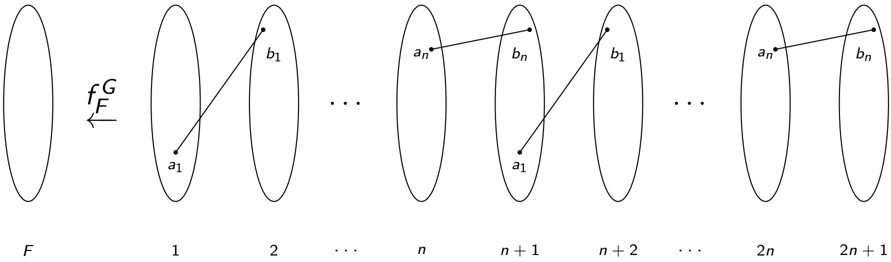
*If for every graph $F \in \mathcal{G}$ there is a graph $G \in \mathcal{G}$ and a confluent epimorphism $f^G_F \colon G \to F$ such that for every two connected graphs $A, B \subseteq G$ such that $G = A \cup B$ we have $f^G_F(A) = F$ or $f^G_F(B) = F$. Then $\mathbb{G}$ is an indecomposable topological graph.*

**Corollary**

*The continuum $|\mathbb{G}|$ is indecomposable.*

# Indecomposable continuum 2

$(\langle a_i, b_i \rangle)_{i \leq n}$ enumerate all edges of $F$
For every two connected graphs $A, B \subseteq G$ such that $G = A \cup B$ we have $f_F^G(A) = F$ or $f_F^G(B) = F$.



# Main theorem - part 2: Embedding solenoids and non-homogeneity

**Theorem (Charatonik-K-Roe '22+)**

*There is a dense set of points in $|\mathbb{G}|$ that belong to a solenoid. Moreover, the only solenoid that embeds into $|\mathbb{G}|$ is the universal solenoid.*

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**Theorem (Charatonik-K-Roe '22+)**

*There is a dense set of points in $|\mathbb{G}|$ that do not belong to a solenoid.*

**Corollary (Charatonik-K-Roe '22+)**

*The continuum $|\mathbb{G}|$ is not homogeneous.*

# Embedding solenoids: lifting cycles

**Lemma**

*Let $A, B \in \mathcal{G}$ and let $f\colon B \to A$ be a confluent epimorphism. Let $C = (c_0, c_1, \ldots, c_n = c_0)$ be a cycle in $A$. Then there is an induced subgraph $D$ of $B$ such that $D$ is a cycle, $f(D) = C$, and $f|_D$ is a wrapping map.*

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We can additionally make sure that:

- The winding number of $f|_D$ is divisible by a given number $m$.
- For every $x \in C$ and any component $P$ of $f^{-1}(x) \cap D$, we have $|P| \geq 2$.

# Embedding solenoids: graph-solenoids

**Definition**

The inverse limit of an inverse sequence of cycles $\{C_n, p_n\}$, where $p_n$ are confluent epimorphisms, is a graph-solenoid if for infinitely many $n$ the winding number of $p_n$ is greater than 1 and for every $x \in V(C_n)$ every component of $p_n^{-1}(x)$ contains at least 2 vertices.

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- We keep lifting cycles in a Fraïssé sequence of $\mathbb{G}$ and we conclude that $\mathbb{G}$ contains a graph-solenoid as a topological subgraph.
- By the result of Hagopian we have to show that the topological realization is homogeneous and that every proper non-degenerate subcontinuum is an arc.

# Almost wrapping maps

## Definition

A surjective homomorphism $g\colon D \to C$, where $C, D$ are cycles with $C = (c_0, c_1, c_2, \dots, c_k = c_0)$ is an almost wrapping map if there is a confluent epimorphism $f\colon D \to C$ such that for every $y \in D$ and $x = c_i \in C$, if $f(y) = c_i$ then $g(y) \in \{c_i, c_{i+1}\}$.

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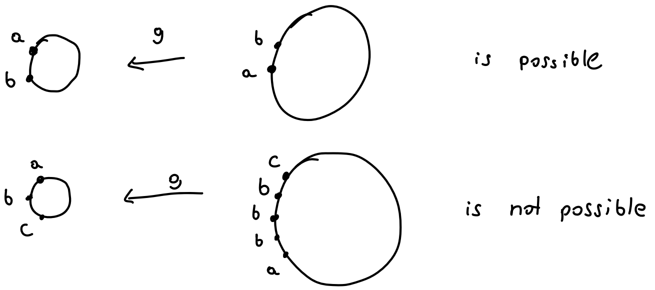

# Lifting solenoids

**Theorem (Charatonik-K-Roe '22+)**

*Let $S$ be a solenoid and let $\mathbb{S}$ be a topological graph whose topological realization is $S$ via a quotient map $\pi_S \colon \mathbb{S} \to S$. Suppose that the set of one-element equivalence classes is dense in $\mathbb{S}$. Then $\mathbb{S}$ is an almost graph-solenoid.*